Examples of multiple solutions for the Yamabe problem on scalar curvature

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Abstract

In the conformal class of a Riemannian metric on a compact connected manifold, there exists at least one metric with constant scalar curvature. In the case with positive scalar curvature, there may be many (non-homothetic) metrics with constant scalar curvature in a conformal class. R. Schoen gave a beautiful example of that phenomenon for a one-parameter family of metrics on $S^1 \times S^{n-1}$. In a preceding paper, we showed that Schoen's construction may be generalized on products $S^1 \times N$ (and other related examples). A unique (ordinary) differential equation, depending only on the dimension, is the key to that construction. Here we give some more details on the solutions of that equation and their behavior in a one-parameter family.

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Key words: Riemannian manifold, scalar curvature, conformal class, Yamabe problem.

1 Introduction

Let (M, g) be a compact connected Riemannian manifold with dimension n (≥ 3). We denote by s_g, μ_g, $V(g)$ the scalar curvature, canonical measure and total volume of the Riemannian metric g. Let $[g]$ be the conformal class of g, i.e. $[g]$ is the set of metrics $h^{-2}g$, where h is any smooth positive function on M. The general "Yamabe problem" is the following: *describe all such functions h on M such that the scalar curvature $s(h^{-2}g)$ is constant.*

Thanks to the works of many authors, we know that there exists at least one solution to the Yamabe problem. More precisely, we consider the Einstein-Hilbert functional σ on the set $Riem(M)$ of all Riemannian metrics on M:

$$\sigma(g) = \frac{\int_M s_g \mu_g}{[V(g)]^{\frac{n-2}{n}}}.$$

Notice that the power $\frac{n-2}{n}$ is chosen such that $\sigma(k^{-2}g) = \sigma(g)$ for any positive constant k.

The functional σ is bounded from below when restricted to the conformal class $[g]$ of the metric g. The Yamabe constant

$$\mu(g) = \inf_{h>0} \sigma(h^{-2}g)$$

is the infimum of σ in the conformal class $[g]$. So $\mu(g)$ depends only on $[g]$ and may be written $\mu([g])$. A theorem, due to successive works by Yamabe [ ], Trudinger [ ], Aubin [ ], Schoen [ ], asserts that the infimum $\mu([g])$ is always achieved in any conformal class and that the resulting metric $h^{-2}g$ has constant scalar curvature precisely $\mu([g])$ (if normalized such that its volume be one). Furthermore, in the cases where $\mu([g])$ is non-positive, the above solution is unique, and the "Yamabe problem" is solved. (Some survey references for all these results are LBB [Bourbaki], Besse [Einstein], ...).

On the other hand, if $\mu([g])$ is positive, then there may exist many (non homothetic) solutions to the general Yamabe problem. This is currently a very active subject, with contributions by many authors (...). Notice that the constant scalar curvature is positive for all those solutions.

A complete solution of the Yamabe problem is known only for some particular cases, such as Einstein metrics.

R. Schoen has studied in [ ] the particular case of the product $S^1 \times S^{n-1}$, with the one-parameter family of metrics (with volume one) given by the canonical metrics on both factors, but with different scaling factors. In that case, he was able to give a complete answer to the Yamabe problem, with an (almost) explicit description of all solutions.

In a preceding paper ([ ]), we showed that R. Schoen's construction may be generalized to $S^1 \times N$ (and some other related examples) in order to give metrics with multiple solutions for the Yamabe problem. In this paper, we give some more details on the corresponding solutions, and a complete explicit description in dimension 4.

We will recall the basic construction for the product $S^1 \times N$, which reduces our problem to the study of a unique ordinary differential equation (depending only on the dimension n), with boundary conditions depending only on one parameter (given by N). Then we give some of the details of our computations (including some open questions).

2 Statement of the result

We will consider in this paper only the simplest case of the construction that we considered in our previous paper [ ], and we refer to it for some motivations about these studies. On the other hand, we will recall the basic hypothesis with some details, so this paper may be read independently.

Let (N, g_N) be a compact connected Riemannian manifold with dimension $n-1$ (with $n \geq 3$). We assume that the scalar curvature s_N of g_N is a positive constant. In order to simplify the computations, we choose $s_N = (n-1)(n-2)$, which is the value of the (constant) scalar curvature of the canonical sphere S^{n-1}. We denote by V_N the volume of g_N. Notice that $\sigma(g_N) = (n-1)(n-2)V_N^{\frac{2}{n-1}}$. In the computations below, the only relevant parameter will be V_N.

Now we consider the product manifold $M = S^1 \times N$, which is obviously compact, connected and with dimension n. On M, we consider the one-parameter family of metrics g_l which are Riemannian products of the metrics with total length l on S^1 with the metric g_N on N. Notice that $s_{g_l} = (n-1)(n-2)$ and $V(g_l) = V_N l$, hence $\sigma(g_l) = (n-1)(n-2)V_N^{\frac{2}{n}} l^{\frac{2}{n}}$.

We consider the subset C_l of the conformal class $[g_l]$ of g_l given by the metrics $g(l,h) = h^{-2} g_l$, where h is a positive smooth function on M depending only on the first factor S^1, so h may be considered as a function on S^1.

For a given positive l, we choose a parameter t on S^1 through the identification $S^1 = \mathbb{R}/l\mathbb{Z}$. Now $h(t)$ is a positive smooth function on $\mathbb{R}$ which is periodic and such that l is a period (not necessarily the smallest positive period). With such a parameter t, one may write $g_l = dt^2 + g_N$. Then elementary computations give that the scalar curvature of $g(l,h) = h^{-2} g_l$ is given by

$$s_g = (n-1)(n-2)h^2 + (n-1)(2hh'' - nh'^2),$$

and that the volume of $g(l,h)$ is given by $V_g = (\int_0^l [h(t)]^{-n} dt) V_N$.

So $g(l,h)$ has a constant scalar curvature $s_g = n(n-1)$ if and only if h satisfies the (ordinary) differential equation

$$(n-2)h^2 + 2hh'' - nh'^2 = n. \tag{1}$$

Hence the Yamabe problem for the restricted class C_l reduces to

(a) find all solutions h of (1) admitting l as a period,

(b) compute $V_{g(l,h)}$ for these solutions.

Notice that for such a restricted problem the Riemannian manifold (N, g_N) gives only the parameter $V_N = [\frac{\sigma(g_N)}{(n-1)(n-2)}]^{\frac{n-1}{2}}$.

Since g_l has constant scalar curvature, equation (1) has an "obvious" constant solution

$$h_0 = \left(\frac{n}{n-2}\right)^{\frac{1}{2}},$$

and the corresponding metric $g_0 = g(l, h_0)$ satisfies $s_{g_0} = n(n-1)$ and $V(g_0) = \left(\frac{n-2}{n}\right)^{\frac{n}{2}} V_N l$.

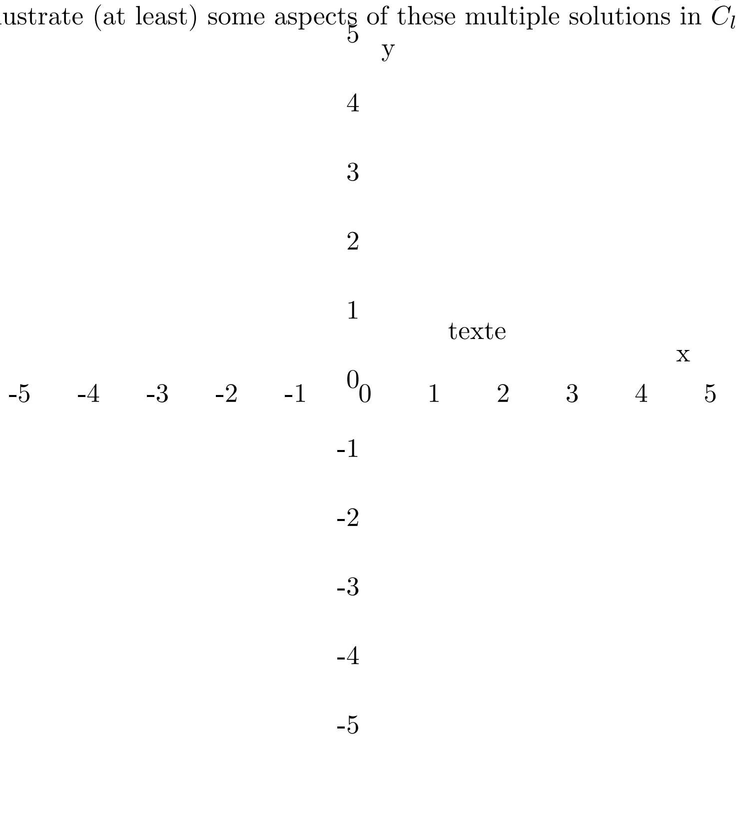
We may now state our results.

Theorem. *For any integer $n \geq 3$, there exists a positive smooth function $f_n : [\frac{2\pi}{\sqrt{n-2}}, +\infty[\to \mathbb{R}$ with the following properties:*
(a) if $0 < l \leq \frac{2\pi}{\sqrt{n-2}}$, equation (1) admits no non-constant function as solution with l as a period;
(b) for any integer $p \geq 1$, and any l with $p\frac{2\pi}{\sqrt{n-2}} < l \leq (p+1)\frac{2\pi}{\sqrt{n-2}}$, equation (1) admits exactly p one-parameter families of non-constant periodic solutions, with smallest period $\frac{1}{k}l$, for all $k = 1, ..., p$;
(c) for p and k as in (b), all the periodic solutions may be written $h_k^c(t) = h_k(\varepsilon t + c)$, where $\varepsilon \in \{-1, 1\}$ and $c \in [0, l[$, and $h_k(t)$ is the unique solution of equation (1) with period exactly $\frac{1}{k}l$, and minimal value at $t = 0$;
(d) on M, $g(l, h_k^c)$ is isometric to $g(l, h_k)$;
(e) all metrics $g(l, h_k^a)$ have constant scalar curvature $n(n-1)$;
(f) $V(g(l, h_k^c)) = V_N k f_n(\frac{l}{k})$;
(g) $f_n(\frac{2\pi}{\sqrt{n-2}}) = \left(\frac{n-2}{n}\right)^{\frac{n}{2}} \frac{2\pi}{\sqrt{n-2}} = V(g(0, \frac{2\pi}{\sqrt{n-2}}))$ and $f_n'(\frac{2\pi}{\sqrt{n-2}}) = 1$;
(h)

$$\lim_{l \to \infty} f_n(l) = \begin{cases} \frac{\pi (2p)!}{2^{2p}(p!)^2} & \text{if } n = 2p+1 \\ \frac{2^{2p-1}((p-1)!)^2}{(2p-1)!} & \text{if } n = 2p \end{cases} ;$$

(i) the coefficient $c_n = \frac{\sqrt{n-2}}{2\pi}(\frac{n}{n-2})^{\frac{n}{2}} \lim_{l \to \infty} f_n(l)$ satisfies the following properties:
(i_1) c_n is strictly increasing if n goes to infinity;
(i_2) $\lim_{n \to \infty} c_n = \frac{e}{\sqrt{2\pi}}$;
(i_3) $c_3 = \frac{3\sqrt{3}}{4}$ (hence $1 < \frac{e}{\sqrt{2\pi}} < c_n \leq \frac{3\sqrt{3}}{4} < 1,3$);
(i_4) $\lim_{l \to \infty} \frac{V(g(l, h_k))}{V(g(\frac{2\pi}{\sqrt{n-2}}, h_0))} = k c_n$, $\forall k \geq 1$;
(j) for $n = 4$, $f_4(l)$ is strictly increasing in $l \in [\pi, +\infty[$.

Remark. (1) We believe that $f_n(l)$ is strictly increasing on $[\frac{2\pi}{\sqrt{n-2}}, +\infty[$ for any $n \geq 3$, but we are able to prove it only for $n = 4$. With a computer, we get also some "evidence" for small n, but we were unable to prove the general case.
(2) In the picture (fig. 1), we indicate the values of $V(g(l, h_p))$ for $0 \leq p \leq [\frac{\sqrt{n-2}}{2\pi} l]$, in order to illustrate (at least) some aspects of these multiple solutions in C_l.



3 General solutions of equation (1)

From now on we will look only for non-constant periodic solutions h of equation (1). After multiplication by $h^{-n-1}h'$ the differential equation (1) becomes $(n-2)h^{1-n}h' + (h^{-n}h'^2)' = nh^{-n-1}h'$, or

$$(-h^{2-n} + h^{-n}h'^2 + h^{-n})' = 0.$$

So, for any solution h, there exists a constant K such that $-h^{2-n} + h^{-n}h'^2 + h^{-n} + K = 0$, that is

$$h'^2 = -Kh^n + h^2 - 1. \tag{2}$$

For $K = 0$ the equation becomes $h'^2 = h^2 - 1$ whose solutions $h(t) = \cosh(t + c)$ are not periodic. Let $K \neq 0$ and consider

$$F_K(s) = -Ks^n + s^2 - 1.$$

For $K < 0$ the derivative ${F_K}'(s) = -nKs^{n-1} + 2s$ is positive for $s > 0$, so equation (2) has no periodic solution.

Now let $K > 0$. Then the derivative ${F_K}'$ vanishes at the point $s_1 = (\frac{2}{nK})^{\frac{1}{n-2}}$. The function F_K increases from -1 to $F_K(s_1)$ on the interval $[0, s_1]$ and decreases from $F_K(s_1)$ to $-\infty$ on the interval $[s_1, +\infty[$. We have

$$F_K(s_1) > 0 \Leftrightarrow K < \frac{2}{n}\left(\frac{n-2}{n}\right)^{\frac{n-2}{2}}.$$

So if $K \geq \frac{2}{n}\left(\frac{n-2}{n}\right)^{\frac{n-2}{2}}$, we have no solution of equation (2).

Let $K < \frac{2}{n}\left(\frac{n-2}{n}\right)^{\frac{n-2}{2}}$. Then the function F_K vanishes at two points $0 < s_0 < s_m$, and is positive on the interval $]s_0, s_m[$. Equation (2) is now equivalent to

$$\frac{dh}{\sqrt{F_K(h)}} = \pm dt,$$

that is $\eta_K(h) = \pm t + c$, where

$$\eta_K(h) = \int_{s_0}^{h} \frac{ds}{\sqrt{F_K(s)}}.$$

The function η_K is well defined on the interval $[s_0, s_m]$ since s_0 and s_m are simple roots of the polynomial F_K. We have $\eta_K'(h) = \frac{1}{\sqrt{F_K(h)}}$, so the function η_K increases from 0 to $\eta_K(s_m)$ on the interval $[s_0, s_m]$ and its derivatives at s_0 and s_m are $+\infty$. Hence η_K admits an inverse function η_K^{-1} which increases from s_0 to s_m on the interval $[0, \eta_K(s_m)]$ and whose derivatives at 0 and $\eta_K(s_m)$ vanish. Equation (2) becomes

$$h(t) = \eta_K^{-1}(\pm t + c).$$

It follows that the positive periodic solutions of equation (2) are the functions $h_{K,c} : t \mapsto h_K(t + c)$ where h_K is the even, $2\eta_K(s_m)$-periodic function given by $h_K(t) = \eta_K^{-1}(t)$ for $t \in [0, \eta_K(s_m)]$. We are looking for solutions having period l, that is for functions whose smallest positive period is of the form l/k where k is a positive integer. So the non-constant solutions of our problem are the functions $h_{K,c}$ for which there exists a positive integer k such that

$$\eta_K(s_m) = \frac{l}{2k}. \tag{3}$$

For such solution $h_{K,c}$ we have

$$\begin{aligned}
V({h_{K,c}}^{-2}g_l) &= V_N \int_0^{2k\eta_K(s_m)} h_K(t)^{-n}dt \\
&= 2kV_N \int_0^{\eta_K(s_m)} h_K(t)^{-n}dt \\
&= \frac{V_N}{\eta_K(s_m)} \int_0^{\eta_K(s_m)} \frac{h_K(t)^{-n}h_K'(t)}{\sqrt{F_K(h_K(t))}}dt \\
&= \frac{V_N}{\eta_K(s_m)} \int_{s_0}^{s_m} \frac{t^{-n}}{\sqrt{F_K(t)}}dt.
\end{aligned}$$

Let $x = K^{\frac{1}{n-2}}s$, $a = K^{\frac{1}{n-2}}s_0$ and $b = K^{\frac{1}{n-2}}s_m$. Then $F_K(s) = K^{-\frac{2}{n-2}}(-x^n + x^2 - K^{\frac{2}{n-2}})$, so we get

$$\eta_K(s_m) = \int_a^b \frac{dx}{\sqrt{-x^n + x^2 - K^{\frac{2}{n-2}}}}$$

and

$$\int_a^b \frac{t^{-n}}{\sqrt{F_K(t)}}\,dt = K^{\frac{n}{n-2}} \int_a^b \frac{x^{-n}dx}{\sqrt{-x^n + x^2 - K^{\frac{2}{n-2}}}}.$$

Clearly a and b are the roots of the polynomial $-x^n + x^2 - K^{\frac{2}{n-2}}$, so

$$K = (a^2 - a^n)^{\frac{n-2}{2}} = (b^2 - b^n)^{\frac{n-2}{2}}.$$

The function $a \mapsto K$ is an increasing bijection from $]0, r[$ to $]0, \frac{2}{n}\left(\frac{n-2}{n}\right)^{\frac{n-2}{2}}[$, where

$$r = \left(\frac{2}{n}\right)^{\frac{1}{n-2}}.$$

So, the condition $0 < K < \frac{2}{n}\left(\frac{n-2}{n}\right)^{\frac{n-2}{2}}$ is equivalent to $0 < a < r$.

For $a \in]0, r[$ we define the following functions

$$P_a(x) = -x^n + x^2 + a^n - a^2,$$

$$\varphi(a) = \int_a^b \frac{dx}{\sqrt{P_a(x)}},$$

$$\psi(a) = (a^2 - a^n)^{\frac{n}{2}} \int_a^b \frac{x^{-n}dx}{\sqrt{P_a(x)}}.$$

Condition (3) becomes

$$\varphi(a) = \frac{l}{2k}$$

and we get

$$V({h_{K,c}}^{-2}g_l) = V_N \frac{\psi(a)}{\varphi(a)} l = 2kV_N\psi(a).$$

4 Variations of the function φ

We have the factorization

$$P_a(x) = (x-a)(b-x)Q_a(x),$$

with

$$Q_a(x) = x^{n-2} + (b+a)x^{n-3} + (b^2+ba+a^2)x^{n-4} + ...$$

$$+(b^{n-3} + b^{n-4}a + ... + a^{n-3})x + (b^{n-2} + b^{n-3}a + ... + a^{n-2} - 1)$$

$$= \sum_{k=1}^{n-1} \frac{b^k - a^k}{b-a} x^{n-1-k} - 1.$$

Let $x = a\cos^2 t + b\sin^2 t$. Then $(x-a)(b-x) = (b-a)^2 \sin^2 t \cos^2 t$, and $dx = (b-a)\sin 2t dt$, so we get

$$\varphi(a) = 2\int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{Q_a(a\cos^2 t + b\sin^2 t)}}.$$

We have $\varphi(r) = \frac{\pi}{\sqrt{Q_r(r)}}$, and $Q_r(r) = [1+2+...+(n-1)]r^{n-2} - 1 = n-2$, so

$$\varphi(r) = \frac{\pi}{\sqrt{n-2}}.$$

Using Fatou's lemma we get $\lim_{a\to 0} \varphi(a) \geq 2\int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{Q_0(\sin^2 t)}}$; we have $Q_0(x) = x\frac{1-x^{n-2}}{1-x}$, so $\sqrt{Q_0(\sin^2 t)} = \sin t \frac{\sqrt{1-\sin^{2n-4} t}}{\cos t}$, which shows that

$$\lim_{a\to 0} \varphi(a) = +\infty.$$

The constant term of the polynomial $Q_a(x)$ is $Q_a(0) = \frac{-P_a(0)}{ab} > 0$, so all the coefficients of the polynomial $Q_a(x)$ are positive. Moreover:

Lemma 1. *The coefficients of the polynomial $Q_a(x)$ are increasing functions of $a \in [0,r]$.*

Proof. We have to show that for every $k \in \{2,3,...,n-1\}$, the function

$$F : a \mapsto \frac{b^k - a^k}{b-a}$$

is increasing on $[0,r]$. We denote the derivative of b as a function of a by b'. We have

$$F'(a) = \frac{(b-a)(kb^{k-1}b' - ka^{k-1}) - (b^k - a^k)(b'-1)}{(b-a)^2},$$

so

$$F'(a) > 0 \iff [(k-1)b^k - kab^{k-1} + a^k]b' + [(k-1)a^k - ka^{k-1}b + b^k] > 0$$

$$\iff \left[(k-1) - k\frac{a}{b} + \left(\frac{a}{b}\right)^k\right] b' + \left[(k-1)\left(\frac{a}{b}\right)^k - k\left(\frac{a}{b}\right)^{k-1} + 1\right] > 0.$$

Let $\alpha = \frac{a}{b}$. It follows from relation $a^2 - a^n = b^2 - b^n$ that $b^{n-2} = \frac{1-\alpha^2}{1-\alpha^n}$; from this we see that the function $\alpha \mapsto a$ is an increasing bijection from $[0,1]$ to $[0,r]$. The preceding condition becomes

$$[(k-1)-k\alpha+\alpha^k]b'+[(k-1)\alpha^k-k\alpha^{k-1}+1]>0.$$

The coefficient of b' is positive; indeed if we set $u(\alpha)=(k-1)-k\alpha+\alpha^k$, then $u(1)=0$ and $u'(\alpha)=k(\alpha^{k-1}-1)<0$ for $0<\alpha<1$. Our condition becomes

$$-b'<\frac{(k-1)\alpha^k-k\alpha^{k-1}+1}{(k-1)-k\alpha+\alpha^k}.$$

We denote the second member by $F_k(\alpha)$ and we will show that for every integer $k\geq 1$ we have

$$\forall\alpha\in]0,1[,F_{k+1}(\alpha)<F_k(\alpha);$$

then it will be sufficient to show that $-b'<F_{n-1}(\alpha)$. We have

$$F_{k+1}(\alpha)<F_k(\alpha)\iff\frac{k\alpha^{k+1}-(k+1)\alpha^k+1}{k-(k+1)\alpha+\alpha^{k+1}}<\frac{(k-1)\alpha^k-k\alpha^{k-1}+1}{(k-1)-k\alpha+\alpha^k}$$

$$\iff 1-\alpha^k-k\alpha^{\frac{k-1}{2}}(1-\alpha)>0 \tag{4}$$

Let $f(\alpha)=1-\alpha^k-k\alpha^{\frac{k-1}{2}}(1-\alpha)$. Then $f'(\alpha)=-k\alpha^{\frac{k-3}{2}}g(\alpha)$ with $g(\alpha)=\alpha^{\frac{k+1}{2}}+\frac{k-1}{2}-\frac{k+1}{2}\alpha$; as $g(1)=0$ and $g'(\alpha)=\frac{k+1}{2}(\alpha^{\frac{k-1}{2}}-1)<0$, we have $g(\alpha)>0$ for $0<\alpha<1$, hence $f'(\alpha)<0$; as $f(1)=0$ we have finally that $f(\alpha)>0$ for $0<\alpha<1$.

Let us show now that $-b'<F_{n-1}(\alpha)$. It follows from relation $b^n-b^2=a^n-a^2$ that the derivative b' satisfies $(nb^{n-1}-2b)b'=na^{n-1}-2a$, so

$$b'=\frac{2a-na^{n-1}}{2b-nb^{n-1}}=\alpha\,\frac{(n-2)\alpha^n-n\alpha^{n-2}+2}{-2\alpha^n+n\alpha^2-(n-2)}.$$

The denominator is negative because $2b-nb^{n-1}=nb(r^{n-2}-b^{n-2})<0$. Hence we get

$$-b'<F_{n-1}(\alpha)\iff\alpha\,\frac{(n-2)\alpha^n-n\alpha^{n-2}+2}{2\alpha^n-n\alpha^2+(n-2)}<\frac{(n-2)\alpha^{n-1}-(n-1)\alpha^{n-2}+1}{\alpha^{n-1}-(n-1)\alpha+(n-2)}$$

$$\iff -(n-2)(\alpha-1)^2[1-\alpha^{2n-2}-(n-1)\alpha^{n-2}(1-\alpha^2)]<0.$$

Comparing with (4) we see that the expression between square brackets is positive, which ends the proof of lemma 1.

Proposition 1. *The function φ is strictly decreasing on the interval $]0,r[$.*

Proof. We have

$$\varphi(a)=2\int_0^{\frac{\pi}{4}}\left(\frac{1}{\sqrt{Q_a(a\cos^2t+b\sin^2t)}}+\frac{1}{\sqrt{Q_a(a\sin^2t+b\cos^2t)}}\right)dt.$$

Hence it is enough to show that for all $t\in]0,\frac{\pi}{4}[$, the function

$$a\mapsto\frac{1}{\sqrt{Q_a(a\cos^2t+b\sin^2t)}}+\frac{1}{\sqrt{Q_a(a\sin^2t+b\cos^2t)}}$$

is decreasing on $]0,r[$. For fixed $t\in]0,\frac{\pi}{4}[$ we set

$$u(a)=Q_a(a\cos^2t+b\sin^2t),$$
$$v(a)=Q_a(a\sin^2t+b\cos^2t).$$

It follows from the fact that the coefficients of the polynomial $Q_a(x)$ are positive, that $0<u<v$. From lemma 1 we have in particular that $b'>-1$, so

$$(a\cos^2 t + b\sin^2 t)' = \cos^2 t + b'\sin^2 t > \cos^2 t - \sin^2 t > 0,$$

and it follows from lemma 1 that $u' > 0$. We will show that $(u+v)' > 0$. Then we get $(uv)' = u(v' + \frac{v}{u}u') > u(v'+u') > 0$, so uv is increasing, and $\frac{2}{\sqrt{uv}}$ is decreasing; moreover

$$\left(\frac{1}{u} + \frac{1}{v}\right)' = -\frac{1}{v^2}\left(\frac{v^2}{u^2}u' + v'\right) < 0,$$

so $\frac{1}{u} + \frac{1}{v}$ is decreasing; now $(\frac{1}{\sqrt{u}} + \frac{1}{\sqrt{v}})^2 = \frac{1}{u} + \frac{1}{v} + \frac{2}{\sqrt{uv}}$ is decreasing, so $\frac{1}{\sqrt{u}} + \frac{1}{\sqrt{v}}$ is decreasing.

Let us show that $(u+v)' > 0$. Let $a = \alpha b$, $\theta = \alpha\cos^2 t + \sin^2 t$, $\eta = \cos^2 t + \alpha \sin^2 t$. Then

$$u + v = \frac{1}{\sin^2 t\cos^2 t}\frac{(1+\alpha)(\alpha^n - \theta^n + 1 - \eta^n)}{(1-\alpha)(1-\alpha^n)} - \frac{1+\cos(2t)}{\cos^2 t},$$

so it is enough to show that the function

$$f_t : \alpha \mapsto \frac{(1+\alpha)(\alpha^n - \theta^n + 1 - \eta^n)}{(1-\alpha)(1-\alpha^n)}$$

is increasing. Straightforward calculation gives the following

$$f_t'(\alpha) = \frac{F_\alpha(t)}{(1-\alpha)^2(1-\alpha^n)^2}$$

where

$$F_\alpha(t) = -2\alpha^{2n} - 2n\alpha^{n+1} + 2n\alpha^{n-1} + 2$$
$$+(\theta^n + \eta^n)(n\alpha^{n+1} + 2\alpha^n - n\alpha^{n-1} - 2) + n(\theta^{n-1}\theta' + \eta^{n-1}\eta')(-\alpha^{n+2} + \alpha^n + \alpha^2 - 1).$$

We have to show that $F_\alpha(t) > 0$ for every $\alpha \in]0,1[$ and every $t \in]0, \frac{\pi}{4}[$. For $t = 0$ we have $\theta = \alpha$, $\eta = 1$, $\theta' = 1$, $\eta' = 0$, and so

$$F_\alpha(0) = 0.$$

Hence it is enough to show that $F_\alpha'(t) > 0$ for $t \in]0, \frac{\pi}{4}[$. We get

$$F_\alpha'(t) = -n(1-\alpha)^2 \sin(2t) G_\alpha(t)$$

where

$$G_\alpha(t) = [(n-1)\alpha^n + n\alpha^{n-1} + 1](\theta^{n-1} - \eta^{n-1})$$
$$+(n-1)(1+\alpha)(1-\alpha^n)(\theta^{n-2}\cos^2 t - \eta^{n-2}\sin^2 t);$$

but

$$\theta^{n-1} - \eta^{n-1} = \theta^{n-2}(\alpha\cos^2 t + \sin^2 t) - \eta^{n-2}(\cos^2 t + \alpha \sin^2 t)$$

$$= \alpha(\theta^{n-2}\cos^2 t - \eta^{n-2}\sin^2 t) + (\theta^{n-2}\sin^2 t - \eta^{n-2}\cos^2 t),$$

so

$$G_\alpha(t) = \theta^{n-2}(A\cos^2 t + B\sin^2 t) - \eta^{n-2}(A\sin^2 t + B\cos^2 t),$$

with

$$A = \alpha^n - n\alpha + n - 1,\ B = (n-1)\alpha^n + n\alpha^{n-1} + 1.$$

We have to show that $G_\alpha(t) < 0$. We have

$$G'_\alpha(t) = \sin(2t)\ G_{\alpha,1}(t),$$

with

$$G_{\alpha,1}(t) = (B-A)(\theta^{n-2} + \eta^{n-2})$$
$$+(n-2)(1-\alpha)[\theta^{n-3}(A\cos^2 t + B\sin^2 t) + \eta^{n-3}(A\sin^2 t + B\cos^2 t)].$$

We have

$$G'_{\alpha,1}(t) = (n-2)(1-\alpha)\sin(2t)\ G_{\alpha,2}(t),$$

with

$$G_{\alpha,2}(t) = 2(B-A)(\theta^{n-3} - \eta^{n-3})$$
$$+(n-3)(1-\alpha)[\theta^{n-4}(A\cos^2 t + B\sin^2 t) - \eta^{n-4}(A\sin^2 t + B\cos^2 t)].$$

For $p \in \{1, 2, ..., n-1\}$, let

$$G_{\alpha,p}(t) = p(B-A)(\theta^{n-1-p} + (-1)^{p+1}\eta^{n-1-p})$$
$$+(n-1-p)(1-\alpha)[\theta^{n-2-p}(A\cos^2 t + B\sin^2 t) + (-1)^{p+1}\eta^{n-2-p}(A\sin^2 t + B\cos^2 t)].$$

Then

$$G'_{\alpha,p}(t) = (n-1-p)(1-\alpha)\sin(2t)\ G_{\alpha,p+1}(t).$$

Clearly $G_\alpha(\frac{\pi}{4}) = 0$, and for even p we have $G_{\alpha,p}(\frac{\pi}{4}) = 0$. We have the following

Lemma 2. *For every $\alpha \in]0,1[$ we have*
a) If $n = 3$, then $G_{\alpha,1}(0) > 0$.
If $n = 4$, then $G_{\alpha,1}(0) = 0$.
If $n \geq 5$, and if p is odd, $p \leq n-2$, then $G_{\alpha,p}(0) < 0$.
b) $G_{\alpha,1}(\pi/4) > 0$.
If p is odd, $\frac{n}{2} \leq p \leq n-1$, then $G_{\alpha,p}(\pi/4) < 0$.
If p is odd, $3 \leq p \leq n-1$, then $[G_{\alpha,p}(\pi/4) \geq 0 \Rightarrow G_{\alpha,p-2}(\pi/4) > 0]$.
c) $G_\alpha(0) < 0$.

Proof. a) If p is odd, we have

$$G_{\alpha,p}(0) = p(B-A)(\theta^{n-1-p} + 1) + (n-1-p)(1-\alpha)(\theta^{n-2-p}A + B)$$

$$= (n-1)[(p-1)\alpha^{2n-1-p} + (p+1)\alpha^{2n-2-p} - (n-1-p)\alpha^{n+1} + (p-1)\alpha^n + n\alpha^{n-1}$$
$$-n\alpha^{n-p} - (p-1)\alpha^{n-1-p} + (n-1-p)\alpha^{n-2-p} - (p+1)\alpha - (p-1)],$$

in particular

$$G_{\alpha,1}(0) = \alpha(n-1)[2\alpha^{2n-4} - (n-2)\alpha^n + (n-2)\alpha^{n-4} - 2].$$

For $n = 3$ we get $G_{\alpha,1}(0) = 2(1-\alpha^2)(1-\alpha)^2 > 0$. For $n = 4$ we get $G_{\alpha,1}(0) = 0$. Let now $n \geq 5$. First we show that $G_{\alpha,1}(0) < 0$. We have

$$G_{\alpha,1}(0) = \alpha(n-1)f(\alpha)$$

with

$$f(\alpha) = 2\alpha^{2n-4} - (n-2)\alpha^n + (n-2)\alpha^{n-4} - 2.$$

We get

$$f'(\alpha) = (n-2)\alpha^{n-5}h(\alpha)$$

with

$$h(\alpha) = 4\alpha^n - n\alpha^4 + n - 4.$$

We have $h(1) = 0$ and $h'(\alpha) = 4n\alpha^3(\alpha^{n-4} - 1) < 0$, so $h(\alpha) > 0$ and f is increasing; as $f(1) = 0$, we have $f(\alpha) < 0$ and $G_{\alpha,1}(0) < 0$.
Let $G_{\alpha,p}(0) = (n-1)h_{n,p}(\alpha)$, then

$$h_{n+1,p+1}(\alpha) - h_{n,p}(\alpha) = p\ u(\alpha) + n\ v(\alpha) + w(\alpha)$$

with

$$u(\alpha) = \alpha^{2n-p} - \alpha^{2n-2-p} + \alpha^{n+2} - \alpha^n = (\alpha^2 - 1)(\alpha^{2n-2-p} + \alpha^n) < 0,$$

$$v(\alpha) = -\alpha^{n+2} + \alpha^{n+1} + \alpha^n - \alpha^{n-1} = \alpha^{n-1}(1-\alpha)(\alpha^2 - 1) < 0,$$

$$\begin{aligned} w(\alpha) &= 3\alpha^{2n-1-p} - \alpha^{2n-2-p} + \alpha^{n+2} - \alpha^{n+1} + 2\alpha^n - \alpha^{n-p} - \alpha^{n-1-p} - \alpha - 1 \\ &= (\alpha-1)(\alpha^{2n-2-p} + \alpha^{n+1}) + (\alpha^{n-1} - 1)(\alpha^{n-p} + \alpha) + (\alpha^n - 1)(\alpha^{n-1-p} + 1) < 0, \end{aligned}$$

which implies that $h_{n+1,p+1}(\alpha) < h_{n,p}(\alpha)$. We know that $h_{n,1}(\alpha) = \frac{G_{\alpha,1}(0)}{n-1} < 0$ for all $n \geq 5$, $h_{4,1}(\alpha) = 0$, and we have $h_{4,2}(\alpha) = 4\alpha(\alpha+1)(\alpha^2 - 1) < 0$, which ends the proof of a).

b) If p is odd, we get

$$G_{\alpha,p}(\pi/4) = (n-1)\frac{(1+\alpha)^{n-1-p}}{2^{n-2-p}}f(\alpha)$$

with

$$f(\alpha) = -(n-2p)\alpha^n + n\alpha^{n-1} - n\alpha + n - 2p.$$

We have

$$f'(\alpha) = n[-(n-2p)\alpha^{n-1} + (n-1)\alpha^{n-2} - 1],$$

$$f''(\alpha) = n(n-1)\alpha^{n-3}[-(n-2p)\alpha + n - 2].$$

For $p = 1$ we have $f''(\alpha) = n(n-1)(n-2)\alpha^{n-3}(1-\alpha) > 0$ and $f'(1) = 0$, so $f' < 0$ on $]0,1[$; as $f(1) = 0$, we get $f > 0$ on $]0,1[$, that is $G_{\alpha,1}(\pi/4) > 0$.

For $p = n/2$ (in case where $n/2$ is odd) we have $f'(\alpha) = n[(n-1)\alpha^{n-2} - 1]$, so f' vanishes at $\alpha_0 = (n-2)^{-\frac{1}{n-2}}$ and f is decreasing on $[0, \alpha_0]$ and increasing on $[\alpha_0, 1]$; as $f(0) = f(1) = 0$ it follows that $f < 0$ on $]0,1[$, so $G_{\alpha,p}(\pi/4) < 0$.

For $p > n/2$ we have $f'' > 0$ on $]0,1[$; as $f'(0) = -n$ and $f'(1) = 2n(p-1)$, there exists $\alpha_0 \in]0,1[$ such that f is decreasing on $[0, \alpha_0]$ and increasing on $[\alpha_0, 1]$; as $f(0) = n - 2p < 0$ and $f(1) = 0$ it follows that $f < 0$ on $]0,1[$, so $G_{\alpha,p}(\pi/4) < 0$.

Now let us suppose that $G_{\alpha,p}(\pi/4) \geq 0$. We have

$$f(\alpha) = 2p(\alpha^n - 1) - n\alpha^n + n\alpha^{n-1} - n\alpha + n,$$

and this expression is a strictly increasing function of p, from which we conclude that $G_{\alpha,p-2}(\pi/4) > 0$.

c) We have $G_\alpha(0) = \alpha^{n-2}A - B = \alpha^{2n-2} - (n-1)\alpha^n + (n-1)\alpha^{n-2} - 1$. Denoting this expression by $f(\alpha)$ we get $f'(\alpha) = (n-1)\alpha^{n-3}g(\alpha)$ with $g(\alpha) = 2\alpha^n - n\alpha^2 + n - 2$. As $g(1) = 0$ and $g'(\alpha) = 2n\alpha(\alpha^{n-2} - 1) < 0$, we have $g > 0$ on $]0,1[$, and f is increasing. As $f(1) = 0$ we have $f < 0$ on $]0,1[$, so $G_\alpha(0) < 0$.

In order to achieve the proof of proposition 1 we have to show that $G_\alpha(t) < 0$ for all $t \in]0, \pi/4[$, all $\alpha \in]0,1[$ and all integer $n \geq 3$. We have

$$G_{\alpha,n-1}(t) = (n-1)(B-A)[1+(-1)^n],$$

and

$$B - A = (n-2)(\alpha^n - 1) + n\alpha(\alpha^{n-2} - 1) < 0.$$

Let us consider first the case $n = 3$. Then $G_{\alpha,2}(t) = 0$, so $G_{\alpha,1}(t)$ is constant. By lemma 2 this constant is positive, so $G_\alpha(t)$ is increasing. As $G_\alpha(\pi/4) = 0$, we have $G_\alpha < 0$ on $]0, \pi/4[$.

Now let $n = 4$. Then $G_{\alpha,3}(t) = 6(B-A) < 0$, so $G_{\alpha,2}(t)$ is decreasing. As $G_{\alpha,2}(\pi/4) = 0$, we have $G_{\alpha,2} > 0$ on $]0, \pi/4[$, so $G_{\alpha,1}(t)$ is increasing. As $G_{\alpha,1}(0) = 0$, we have $G_{\alpha,1} > 0$ on $]0, \pi/4[$, so $G_\alpha(t)$ is increasing. As $G_\alpha(\pi/4) = 0$, we have $G_\alpha < 0$ on $]0, \pi/4[$.

Now let $n \geq 5$, n odd. Then $G_{\alpha,n-1} = 0$, so $G_{\alpha,n-2}(t)$ is constant. As $G_{\alpha,n-2}(0) < 0$, this constant is negative. By lemma 2 there exists an odd integer p_0 (depending on α) such that $G_{\alpha,p}(\pi/4) \geq 0$ and such that for every odd integer p we have

$$p_0 < p \leq n-2 \Rightarrow G_{\alpha,p}(\pi/4) < 0$$

and

$$1 \leq p < p_0 \Rightarrow G_{\alpha,p}(\pi/4) > 0.$$

For every odd integer p such that $p_0 < p \leq n-2$ we then have:

$$G_{\alpha,p+1} : \searrow_0, G_{\alpha,p+1} > 0, \;\; G_{\alpha,p} : \nearrow^-, G_{\alpha,p} < 0,$$

so

$$G_{\alpha,p_0+1} : \searrow_0, \;\; G_{\alpha,p_0+1} > 0, \;\; G_{\alpha,p_0} : \;_-\nearrow^+, \;\; G_{\alpha,p_0} : \;[-,+],$$

and for every odd integer p such that $1 \leq p < p_0$ we have:

$$G_{\alpha,p+1} : \searrow \nearrow^0, \;\; G_{\alpha,p+1} : \;[+,-], \;\; G_{\alpha,p} : \;_- \nearrow \searrow_+, \;\; G_{\alpha,p} : \;[-,+],$$

so

$$G_{\alpha,1} : \;[-,+],$$

and finally

$$G_\alpha : \;^- \searrow \nearrow^0,$$

so $G_\alpha < 0$ on $]0, \pi/4[$

For $n \geq 5$, n even, we have $G_{\alpha,n-1} = 2(n-1)(B-A) < 0$, and like before we conclude that $G_\alpha < 0$ on $]0, \pi/4[$.

5 On the function f_n

We have shown in the preceding paragraph that the function φ decreases from $+\infty$ to $\frac{\pi}{\sqrt{n-2}}$ on the interval $]0, r]$. The existence condition (3) for a non-constant solution of our problem stated in paragraph 2 is $\varphi(a) = \frac{l}{2k}$, ($k$ integer ≥ 1).

For every $l > 0$ our problem has the constant solution $h_0 \equiv \sqrt{\frac{n}{n-2}}$. It satisfies

$$V(h_0^{-2} g_l) = \left(\frac{n-2}{n}\right)^{\frac{n}{2}} V_N l.$$

For $l \leq \frac{2\pi}{\sqrt{n-2}}$ this is the only solution of equation (1).

Let $p \in \mathbb{N}^*$ with $p\frac{2\pi}{\sqrt{n-2}} < l \le (p+1)\frac{2\pi}{\sqrt{n-2}}$. Then for any $k = 1, 2, ..., p$, $\frac{l}{2k} \ge \frac{l}{2p} > \frac{\pi}{\sqrt{n-2}}$ and $a = \varphi^{-1}(\frac{l}{2k})$ exists and belongs to $]0, r[$. The solution h_k^l of (1) corresponding to a satisfies

$$V((h_k^l)^{-2}g_l) = V_N k f_n(l/k),$$

where we define the function f_n by

$$f_n(l) = 2\psi(\varphi^{-1}(l/2)). \tag{5}$$

If $l \to \frac{2\pi}{\sqrt{n-2}}$, $(l > \frac{2\pi}{\sqrt{n-2}})$, then $\varphi^{-1}(\frac{l}{2}) \to \varphi^{-1}(\frac{\pi}{\sqrt{n-2}}) = r$, and $f_n(l) \to 2\psi(r) = 2\left(\frac{n-2}{n}\right)^{\frac{n}{2}} \frac{\pi}{\sqrt{n-2}}$, so

$$\lim_{l \to \frac{2\pi}{\sqrt{n-2}}} V(g(l, h_1)) = V\left(g\left(\frac{2\pi}{\sqrt{n-2}}, h_0\right)\right).$$

Proposition 2. *We have $(f_n)'_{right}(\frac{2\pi}{\sqrt{n-2}}) = \left(\frac{n-2}{n}\right)^{\frac{n}{2}}$, which shows that $V(g(l, h_1))$ and $V(g(l, h_0))$ have the same derivative at the point $l = \frac{2\pi}{\sqrt{n-2}}$.*

Proof. For $l > \frac{2\pi}{\sqrt{n-2}}$ we have $f_n'(l) = \frac{\psi'(\varphi^{-1}(\frac{l}{2}))}{\varphi'(\varphi^{-1}(\frac{l}{2}))}$, so

$$(f_n)'_{right}\left(\frac{2\pi}{\sqrt{n-2}}\right) = \lim_{s \to r} \frac{\psi'(s)}{\varphi'(s)}.$$

From $a^2 - b^2 = a^n - b^n$ and $a \ne b$ we get $a + b = \sum_{i=o}^{n-1} a^{n-1-i}b^i$, so

$$1 + b' = \sum_{i=o}^{n-1} [(n-1-i)a^{n-2-i}b^i + ia^{n-1-i}b^{i-1}b'],$$

from which we get $1 + b'(r) = (n-1)[1 + b'(r)]$, so

$$b'(r) = -1.$$

Let $a = r - h$, then $b = r + h + \alpha h^2 + \beta h^3 + \gamma h^4 + ...$ Putting this in relation $a^2 - b^2 = a^n - b^n$, we get the values of the constants α, β, γ:

$$b = r + h - \frac{n-1}{3r}h^2 + \left(\frac{n-1}{3r}\right)^2 h^3 - \frac{(n-1)(19n^2 - 23n + 58)}{540r^3}h^4 + ...$$

Let $x = a\cos^2 t + b \sin^2 t$. Then we get after some calculations

$$Q_a(x) = (n-2)\{1 - \frac{n-1}{3r}\cos 2t \cdot h - \frac{n-1}{36r^2}[12 - 2(n-1)\cos 2t - 3(n-3)\cos^2 2t]h^2\} + ...,$$

$$\frac{1}{\sqrt{Q_a(x)}} = \frac{1}{\sqrt{n-2}}\{1 + \frac{n-1}{6r}\cos 2t \cdot h + \frac{n-1}{72r^2}[3(n+2) - 2(n-1)\cos 2t + 3\cos 4t]h^2\} + ...,$$

$$\varphi(a) = \frac{\pi}{\sqrt{n-2}}\left[1 + \frac{(n-1)(n+2)}{24r^2}h^2\right] + ...,$$

$$\psi(a) = \frac{\pi}{\sqrt{n-2}}\left(\frac{n-2}{n}\right)^{\frac{n}{2}}\left[1 + \frac{(n-1)(n+2)}{24r^2}h^2\right] + ...,$$

so

$$(f_n)'_{right}(\tfrac{2\pi}{\sqrt{n-2}}) = \frac{\psi''(r)}{\varphi''(r)} = \left(\frac{n-2}{n}\right)^{\frac{n}{2}}.$$

Proposition 3. *For $l > \frac{2\pi}{\sqrt{n-2}}$ we have $V(g(l,h_1)) < V(g(l,h_0))$.*

Proof. As $V(g(l,h_1)) = 2V_N\psi(a)$ and $V(g(l,h_0)) = 2V_N\left(\frac{n-2}{n}\right)^{\frac{n}{2}}\varphi(a)$, with $a = \varphi^{-1}(\frac{l}{2})$, we have to show that for all $a \in]0,r[$,

$$\psi(a) < (1-\tfrac{2}{n})^{\frac{n}{2}}\varphi(a).$$

Let us recall that $\varphi(a) = \int_a^b \frac{dx}{\sqrt{P_a(x)}}$ and $\psi(a) = (a^2-a^n)^{\frac{n}{2}}\int_a^b \frac{x^{-n}dx}{\sqrt{P_a(x)}}$. Let $s = \sqrt{\frac{x^2}{a^2-a^n}-1}$, $s_0 = \sqrt{\frac{a^{n-2}}{1-a^{n-2}}}$, $s_m = \sqrt{\frac{b^{n-2}}{1-b^{n-2}}}$; then

$$\begin{aligned}
\varphi(a) &= \int_{s_0}^{s_m} \frac{ds}{(1+s^2)^{\frac{1}{2}}\sqrt{1-(a^2-a^n)^{\frac{n-2}{2}}\frac{(1+s^2)^{\frac{n}{2}}}{s^2}}} \\
&= \sum_{p=0}^{+\infty}\frac{(2p)!}{2^{2p}(p!)^2}(a^2-a^n)^{p\frac{n-2}{2}}\int_{s_0}^{s_m}\frac{(1+s^2)^{\frac{pn-1}{2}}}{s^{2p}}ds, \\
\psi(a) &= \int_{s_0}^{s_m} \frac{ds}{(1+s^2)^{\frac{n+1}{2}}\sqrt{1-(a^2-a^n)^{\frac{n-2}{2}}\frac{(1+s^2)^{\frac{n}{2}}}{s^2}}} \\
&= \sum_{p=0}^{+\infty}\frac{(2p)!}{2^{2p}(p!)^2}(a^2-a^n)^{p\frac{n-2}{2}}\int_{s_0}^{s_m}\frac{(1+s^2)^{\frac{(p-1)n-1}{2}}}{s^{2p}}ds,
\end{aligned}$$

so it is enough to show that for all integer $p \geq 0$ and all $a \in]0,r[$ we have

$$\int_{s_0}^{s_m}\frac{(1+s^2)^{\frac{(p-1)n-1}{2}}}{s^{2p}}ds < (1-\tfrac{2}{n})^{\frac{n}{2}}\int_{s_0}^{s_m}\frac{(1+s^2)^{\frac{pn-1}{2}}}{s^{2p}}ds.$$

Let

$$F_p(a) = (1-\tfrac{2}{n})^{\frac{n}{2}}\int_{s_0}^{s_m}\frac{(1+s^2)^{\frac{pn-1}{2}}}{s^{2p}}ds - \int_{s_0}^{s_m}\frac{(1+s^2)^{\frac{(p-1)n-1}{2}}}{s^{2p}}ds.$$

If $a = r$, then $s_0 = s_m$, so $F(r) = 0$. Hence it is enough to show that the function F is strictly decreasing on $]0,r[$. We get

$$\tfrac{2}{n-2}(a^2-a^n)^{1+\frac{p(n-2)}{2}}F_p'(a) = [(1-\tfrac{2}{n})^{\frac{n}{2}} - (1-b^{n-2})^{\frac{n}{2}}]b^{\frac{n}{2}}b' - [(1-\tfrac{2}{n})^{\frac{n}{2}} - (1-a^{n-2})^{\frac{n}{2}}]a^{\frac{n}{2}},$$

so we have to show that

$$b' < \frac{a^{\frac{n}{2}}}{b^{\frac{n}{2}}}\frac{(1-\frac{2}{n})^{\frac{n}{2}} - (1-a^{n-2})^{\frac{n}{2}}}{(1-\frac{2}{n})^{\frac{n}{2}} - (1-b^{n-2})^{\frac{n}{2}}},$$

i.e.

$$\frac{a}{b}\,\frac{(1-\frac{2}{n}) - (1-a^{n-2})}{(1-\frac{2}{n}) - (1-b^{n-2})} < \left(\frac{a}{b}\right)^{\frac{n}{2}}\frac{(1-\frac{2}{n})^{\frac{n}{2}} - (1-a^{n-2})^{\frac{n}{2}}}{(1-\frac{2}{n})^{\frac{n}{2}} - (1-b^{n-2})^{\frac{n}{2}}}. \tag{6}$$

Let $\alpha = \frac{a}{b}$ and $u = \sqrt{\frac{n}{n-2}\frac{1-\alpha^{n-2}}{1-\alpha^n}}$. Clearly $u > 1$ and $\alpha u < 1$. Moreover we have $\alpha^{1/2}u < 1$. Inequality (5) becomes

$$\frac{1-u^2}{1-(\alpha u)^2} < \alpha^{\frac{n}{2}-1}\frac{1-u^n}{1-(\alpha u)^n},$$

i.e.

$$\left(\sum_{k=0}^{n-1}(\alpha u)^k\right)(1+u)-\alpha^{\frac{n}{2}-1}(1+\alpha u)\left(\sum_{k=0}^{n-1}u^k\right)>0. \tag{7}$$

For even n this inequality reads

$$\left(\sum_{k=0}^{\frac{n-2}{2}}[(\alpha u)^k+(\alpha u)^{n-1-k}]\right)(1+u)-\alpha^{\frac{n}{2}-1}(1+\alpha u)\left(\sum_{k=0}^{\frac{n-2}{2}}(u^k+u^{n-1-k})\right)>0.$$

For each integer k such that $0\le k\le \frac{n-2}{2}$ we have

$$[(\alpha u)^k+(\alpha u)^{n-1-k}](1+u)-\alpha^{\frac{n}{2}-1}(1+\alpha u)(u^k+u^{n-1-k})$$

$$=(\alpha u)^k\left[(1+(\alpha u)^{n-1-2k})(1+u)-\alpha^{\frac{n}{2}-1-k}(1+\alpha u)(1+u^{n-1-2k})\right]$$

$$=(\alpha u)^k\left[(1-\alpha^{\frac{n}{2}-1-k})(1-(\alpha^{1/2}u)^{n-2k})+u(1-\alpha^{\frac{n}{2}-k})(1-(\alpha^{1/2}u)^{n-2-2k})\right],$$

which is positive as $\alpha^{1/2}u<1$. This proves inequality (7) in the case n even. For odd n this inequality reads

$$\left((\alpha u)^{\frac{n-1}{2}}+\sum_{k=0}^{\frac{n-3}{2}}[(\alpha u)^k+(\alpha u)^{n-1-k}]\right)(1+u)-\alpha^{\frac{n}{2}-1}(1+\alpha u)\left(u^{\frac{n-1}{2}}+\sum_{k=0}^{\frac{n-3}{2}}(u^k+u^{n-1-k})\right)>0,$$

i.e.

$$(\alpha u)^{\frac{n-1}{2}}(1+u)-\alpha^{\frac{n}{2}-1}(1+\alpha u)u^{\frac{n-1}{2}}$$

$$+\sum_{k=0}^{\frac{n-3}{2}}\left((\alpha u)^k\left[(1-\alpha^{\frac{n}{2}-1-k})(1-(\alpha^{1/2}u)^{n-2k})+u(1-\alpha^{\frac{n}{2}-k})(1-(\alpha^{1/2}u)^{n-2-2k})\right]\right)>0.$$

The first term is negative:

$$(\alpha u)^{\frac{n-1}{2}}(1+u)-\alpha^{\frac{n}{2}-1}(1+\alpha u)u^{\frac{n-1}{2}}=-\alpha^{\frac{n}{2}-1}u^{\frac{n-1}{2}}(1-\alpha^{1/2})(1-\alpha^{1/2}u),$$

but it is compensated by the first term in the sum:

$$-\alpha^{\frac{n}{2}-1}u^{\frac{n-1}{2}}(1-\alpha^{1/2})(1-\alpha^{1/2}u)+(1-\alpha^{\frac{n}{2}-1})(1-\alpha^{\frac{n}{2}}u^n)$$

$$=(1-\alpha^{1/2})(1-\alpha^{1/2}u)\left[\left(\sum_{j=0}^{n-3}\alpha^{j/2}\right)\left(\sum_{j=0}^{n-1}(\alpha^{1/2}u)^k\right)-\alpha^{\frac{n}{2}-1}u^{\frac{n-1}{2}}\right];$$

as $\alpha^{\frac{n}{2}-1}u^{\frac{n-1}{2}}=\alpha^{(\frac{n-3}{2})/2}(\alpha^{1/2}u)^{\frac{n-1}{2}}$, the expression between brackets is positive, which ends the proof.

Remark about $\mathbf{V(g(l,h_k))}$ Let k be an integer ≥ 1 and let $l>k\frac{2\pi}{\sqrt{n-2}}$. Then $\frac{l}{2k}=\frac{l/k}{2}>\frac{\pi}{\sqrt{n-2}}$, and

$$V(g(l,h_k))=kV(g(\tfrac{l}{k},h_1)). \tag{8}$$

By proposition 4 we have $kV(g(\frac{l}{k},h_1))<kV(g(\frac{l}{k},h_0))=V(g(l,h_0))$, so

$$V(g(l,h_k))<V(g(l,h_0)). \tag{9}$$

If $l\to k\frac{2\pi}{\sqrt{n-2}}$, then $V(g(l,h_k))\to kV(g(\frac{2\pi}{\sqrt{n-2}},h_1))=kV(g(\frac{2\pi}{\sqrt{n-2}},h_0))$, so

$$\lim_{l\to k\frac{2\pi}{\sqrt{n-2}}} V(g(l,h_k)) = V(g(k\frac{2\pi}{\sqrt{n-2}},h_0)). \tag{10}$$

If $l \to +\infty$, then $V(g(l,h_k)) \to 2kV_N\psi(\varphi^{-1}(+\infty))$, so

$$\lim_{l\to+\infty} V(g(l,h_k)) = 2kV_N\psi(0). \tag{10}$$

In particular $V(g(+\infty,h_k)) = kV(g(+\infty,h_1))$.

The following proposition shows that $\psi(0)$ is finite.

Proposition 4. *Let* $\psi_n(a) = (a^2-a^n)^{\frac{n}{2}}\int_a^b \frac{x^{-n}dx}{\sqrt{P_a(x)}}$.

Proof. We write $\psi_n(a)$ in the form

$$\psi_n(a) = (1-a^{n-2})^{\frac{n}{2}}a^n\Big(\int_a^{\sqrt{a}}\frac{x^{-n}dx}{\sqrt{P_a(x)}} + \int_{\sqrt{a}}^r\frac{x^{-n}dx}{\sqrt{P_a(x)}} + \int_r^b\frac{x^{-n}dx}{\sqrt{P_a(x)}}\Big).$$

For $\sqrt{a} \le x \le r$ we have $x^{-n} \le a^{-n/2}$, and as the function $x \mapsto P_a(x)$ is increasing on $[0,r]$, $P_a(x) \ge P_a(\sqrt{a}) = a - a^{n/2} - a^2 + a^n$, so

$$a^n\int_{\sqrt{a}}^r\frac{x^{-n}dx}{\sqrt{P_a(x)}} \le a^{\frac{n-1}{2}}\frac{r-\sqrt{a}}{\sqrt{1-a^{\frac{n}{2}-1}-a+a^{n-1}}},$$

which shows that

$$\lim_{a\to0} a^n\int_{\sqrt{a}}^r\frac{x^{-n}dx}{\sqrt{P_a(x)}} = 0.$$

As $\int_r^b \frac{dx}{\sqrt{P_a(x)}}$ is bounded when $a \to 0$, we have

$$\lim_{a\to0} a^n\int_r^b\frac{x^{-n}dx}{\sqrt{P_a(x)}} = 0.$$

We have $P_a(x) = (x^2-a^2)\big(1-\frac{x^n-a^n}{x^2-a^2}\big)$, and for $a < x < \sqrt{a}$,

$$\tfrac{n}{2}a^{n-2} < \frac{x^n-a^n}{x^2-a^2} < \frac{a^{n/2}-a^n}{a-a^2},$$

so

$$\frac{1}{\sqrt{1-\frac{n}{2}a^{n-2}}} < \frac{1}{\sqrt{1-\frac{x^n-a^n}{x^2-a^2}}} < \frac{1}{\sqrt{1-\frac{a^{\frac{n}{2}-1}-a^{n-1}}{1-a}}}.$$

It follows that

$$\lim_{a\to0}\psi(a) = \lim_{a\to0} a^n\int_a^{\sqrt{a}}\frac{x^{-n}dx}{\sqrt{x^2-a^2}}.$$

Let $t = \sqrt{x^2-a^2}$, then

$$\int_a^{\sqrt{a}}\frac{x^{-n}dx}{\sqrt{x^2-a^2}} = \int_0^{\sqrt{a-a^2}}\frac{dt}{(t^2+a^2)^{\frac{n+1}{2}}},$$

and standard calculations show the announced results.

Remark on $\mathbf{f_n}(+\infty)$ The function $n \mapsto \psi_n(0)$ is decreasing, so we have for all integer $n \ge 3$, $\psi_n(0) \le \psi_3(0) = \pi/4$. Moreover $\lim_{n\to\infty}\psi_n(0) = 0$.

Proposition 5. *Let $n = 4$. Then*

$$\varphi(a) = \frac{\pi\sqrt{2}}{2} \sum_{p=0}^{+\infty} \frac{(4p)!}{2^{6p}(p!)^2(2p)!}(b^2 - a^2)^{2p},$$

$$\psi(a) = \frac{\pi\sqrt{2}}{8} \sum_{p=0}^{+\infty} \frac{(4p)!}{2^{6p}(p!)^2(2p)!} \; \frac{3}{(4p-3)(4p-1)} \; (b^2 - a^2)^{2p},$$

which shows in particular that the function ψ is decreasing, since $b^2 - a^2$ is decreasing.

Proof. In this case we have $a^2 + b^2 = 1$ and $0 < x_0 < r = \frac{1}{\sqrt{2}} < b < 1$, so there exists $\theta \in]0, \frac{\pi}{4}[$ such that $a = \sin\theta$ and $b = \cos\theta$. Let

$$x^2 = a^2 \cos^2 t + b^2 \sin^2 t;$$

then $2xdx = (b^2 - a^2)\sin 2tdt$, $P_a(x) = (x^2 - a^2)(b^2 - x^2) = \frac{1}{4}(b^2 - a^2)^2 \sin^2 2t$, $x^2 = \frac{1}{2}(1 - \cos 2\theta cos2t)$, so

$$\begin{aligned}
\psi(a) &= (a^2 - a^4)^2 \int_a^b \frac{dx}{x^4\sqrt{P_a(x)}} \\
&= \frac{\sqrt{2}}{4}\sin^4 2\theta \int_0^{\pi/2} \frac{dt}{(1 - \cos 2\theta \cos 2t)^{5/2}} \\
&= \frac{\sqrt{2}}{8}(1 - \cos^2 2\theta)^2 \int_0^{\pi} \frac{dt}{(1 - \cos 2\theta \cos t)^{5/2}}.
\end{aligned}$$

Let $s = \cos 2\theta = b^2 - a^2$; then

$$\begin{aligned}
\psi(a) &= \frac{\sqrt{2}}{8}(1 - s^2)^2 \int_0^{\pi} \left(\sum_{p=0}^{+\infty} (-1)^p \frac{(2p+4)!}{3 \cdot 2^{2p+2}(p+2)!p!} s^p \cos^p t \right) dt \\
&= \frac{\sqrt{2}}{8}(1 - s^2)^2 \sum_{p=0}^{+\infty} (-1)^p \frac{(2p+4)!}{3 \cdot 2^{2p+2}(p+2)!p!} s^p \int_0^{\pi} \cos^p t dt \\
&= \frac{\sqrt{2}}{8}(1 - s^2)^2 \sum_{p=0}^{+\infty} \frac{(4p+4)!}{3 \cdot 2^{4p+2}(2p+2)!(2p)!} s^{2p} \int_0^{\pi} \cos^{2p} t dt \\
&= \frac{\sqrt{2}}{8}(1 - s^2)^2 \sum_{p=0}^{+\infty} \frac{(4p+4)!}{3 \cdot 2^{4p+2}(2p+2)!(2p)!} s^{2p} \frac{(2p)!}{2^{2p}(p!)^2}\pi \\
&= \frac{\pi\sqrt{2}}{8}(1 - s^2)^2 \sum_{p=0}^{+\infty} \frac{(4p+4)!}{3 \cdot 2^{6p+2}(2p+2)!(p!)^2} s^{2p} \\
&= \frac{\pi\sqrt{2}}{8} \sum_{p=0}^{+\infty} \frac{(4p)!}{2^{6p}(p!)^2(2p)!} \; \frac{3}{(4p-3)(4p-1)} s^{2p},
\end{aligned}$$

and the formula for $\varphi(x_0)$ is obtained in the same way.

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